§4 Atiyah–Hitchin–Singer complex and Kuranishi model.

§4.1 Atiyah–Hitchin–Singer complex

$$0 \to L^2_3(\Gamma(\Lambda^0 \otimes ad(P)) \xrightarrow{d_A}$$

$$(*) \cdots \qquad L^2_2(\Gamma(\Lambda^1 \otimes ad(P)) \xrightarrow{d_A^+} L^2_1(\Gamma(\Lambda^+ \otimes ad(P))$$

$$\to 0$$

Denote by $\mathbb{H}^i_A(*)$ the i-th cohomology of $(*)$ $(i = 0, 1, 2)$

Thm (Atiyah–Hitchin–Singer)

If $\mathbb{H}^0_A(*) = \mathbb{H}^2_A(*) = 0$, then around $[A]$, the moduli space $\mathcal{M}_P$ is smooth manifold of the expected dimension $(= -\text{index})$

— analoge of Kodaira–Spencer theorem in the theory of deformation of complex structures.

irreducible / reducible connections

Let $A \in \mathcal{A}_P$. Define the stabilizer group Γ_A of $\mathcal{G}_P$ at A

by $\Gamma_A := \{ g \in \mathcal{G}_P \mid g(A) = A \}$

Def.

A connection A is said to be irreducible if Γ_A coincides with the centre of G, and reducible otherwise.

Denote by $\mathcal{A}_P^*$ the set of all irreducible connections.

$\hat{\mathcal{G}}_P := \mathcal{G}_P / C(G)$, $C(G)$: the centre of G

acts freely on $\mathcal{A}_P^*$.

Define $\mathcal{M}_P^* := \{ A \in \mathcal{A}_P^* \mid A : \text{ASD} \} / \hat{\mathcal{G}}_P$

Prop. $G = SU(n), SO(m)$

If A is irreducible, $\mathbb{H}_A^0(+) = 0$

§4.2 Kuranishi model

Thm (Atiyah–Hitchin–Singer)

Let $[A] \in \mathcal{M}_P^*$.

Then there exists a neibourhood U of 0 in $\mathbb{H}^1_A(*)$ such that around $[A]$, there exists a real analytic map $\kappa_A : U \to \mathbb{H}^2_A(*)$ with $\kappa_A(0) = 0$ and the derivative of κ_A is zero, $\mathcal{M}_P^*$ is locally modelled on the zero set of κ_A.

— analogue of Kuranishi's theorem in the deformation of complex structures

Sketch of proof

Let A : ASD instanton, and consider its deformation $A + \alpha$, $\alpha \in \Omega^1(\mathrm{ad}(P))$. We have.

$$F_{A+\alpha} = F_A + d_A \alpha + \frac{1}{2}[\alpha \wedge \alpha]$$

So

$$F^+_{A+\alpha} = 0 \iff d_A^+ \alpha + \frac{1}{2}[\alpha \wedge \alpha]^+ = 0 \quad \text{in } \triangle$$

where $[\alpha \wedge \alpha]^+ = \pi^+([\alpha \wedge \alpha])$.

Define

$$k_A : L^2_2(\Gamma(\Lambda^1 \otimes ad(P))) \to L^2_2(\Gamma(\Lambda^1 \otimes ad(P)))$$

by $k_A(\alpha) = \alpha + (d_A^+)^*\left(G_A\left(\frac{1}{2}[\alpha \wedge \alpha]^+\right)\right)$,

where G_A is the Green operator for Laplacian operator $\Delta_A^+ = d_A^+ \circ (d_A^+)^*$.

This is a linear operator : $L^2_1(\Gamma(\Lambda^+ \otimes ad(P))) \to \operatorname{Im} \Delta_A^+$,

which satisfies $G_A \circ \Delta_A^+ + H_A = Id$,

where H_A is the projection : $L^2_1(\Gamma(\Lambda^+ \otimes ad(P))) \to \mathbb{H}^2{}_A$

Lem α satisfies $\star$ $\Longleftrightarrow$ $d_A^+(k_A(\alpha)) = 0$

and $H_A([\alpha \wedge \alpha]^+) = 0$

Slice : Remember the "tangent space" to the orbit

$\mathcal{G}_P \cdot A$ at A is given by

$$\{ d_A a \mid a \in L^2_3(\Gamma(\Lambda^0 \otimes ad(P))) \}$$

We consider the orthonormal complement of this, namely,

$$\operatorname{Ker} d_A^* = \{ \alpha \in L^2_2(\Gamma(\Lambda^1 \otimes ad(P))) \mid d_A^* \alpha = 0 \}$$

We define the slice at A by

$$S_{A,\varepsilon} := \{ \alpha \in L^2_2(\Lambda^1 \otimes ad(P)) \mid d_A^* \alpha = 0, \|\alpha\|_{L^2_2} < \varepsilon \}$$

and a map $\rho_{A,\varepsilon} : S_{A,\varepsilon} \to \mathcal{A}_{L^2_2} / \mathcal{G}_{L^2_3}$ by

$\alpha \longmapsto [A + \alpha]$.

Lem.

Let $A \in \mathcal{A}^*_{L^2_2}$, Then $\exists \varepsilon > 0$, s.t.

$$S_{A,\varepsilon} \underset{\text{diffeo}}{\cong} \rho_{A,\varepsilon}(S_{A,\varepsilon}) \subset \mathcal{A}^*_{L^2_2} / \hat{\mathcal{G}}_{L^2_3}$$

Kuranishi model

Define

$$S^+_{A,\varepsilon} := \{ \alpha \in S_{A,\varepsilon} \mid \alpha \text{ satisfies } \text{☆} \}$$

Lem

- $K_A(S^+_{A,\varepsilon}) \subset \mathbb{H}^1_A(+)$
- $\alpha \in S^+_{A,\varepsilon} \iff K_A(\alpha) \in \mathbb{H}^1_A(+)$ and $H_A([\alpha \wedge \alpha]^+) = 0$

Lem $\left.\frac{d\kappa_A}{d\alpha}\right|_{\alpha=0}(\varphi) := \lim_{t\to 0} \frac{\kappa_A(t\varphi) - \kappa_A(0)}{t} = \varphi.$

Thus, from the inverse mapping theorem,

$\exists$ a neighbourhood in $\mathbb{H}^1_A(*)$, and a map

$$^{\exists}\kappa_A^{-1} : U \to L^2_2(\Lambda^1 \otimes ad(P))$$

such that U is diffeomorphic to its image.

Define $\psi_A : \kappa_A^{-1}(U) \to \mathbb{H}^2_A(*)$ by

$$\alpha \mapsto H_A([a \wedge \alpha]^+)$$

and $k_A := \psi_A \circ \kappa_A^{-1} : U \to \mathbb{H}^2_A(*)$

consider the zero set of k_A

$$Z(k_A) := \{\alpha \in \mathbb{H}^1_A(*) \mid \|\alpha\|_{L^2_2} < \varepsilon,\ k_A(\alpha) = 0\}$$

Then by κ_A^{-1}

$$Z(k_A) \underset{\text{diffeo}}{\xrightarrow{\kappa_A^{-1}}} V := \kappa_A^{-1}(Z(k_A))$$

$$\subset S^+_{A,\varepsilon}$$

($\because$) For $\alpha \in V$, $\alpha = \kappa_A^{-1}({}^{\exists}\beta)$, $\beta \in Z(k_A)$.

$\Rightarrow H_A([a \wedge \alpha]^+) = 0$ by definition

also $\kappa_A(\alpha) \in \mathbb{H}^1_A(*)$ as $\kappa_A(\alpha) = \kappa_A \circ \kappa_A^{-1}(\beta)$

$= \beta \in \mathbb{H}^1_A(*)$ $\square$)

Since for $\varepsilon \ll 1$,

$$S_{A,\varepsilon} \underset{\text{diffeo}}{\cong} P_{A,\varepsilon}(S_{A,\varepsilon}) \subset \mathcal{A}^+_{L^2_2} / \mathcal{G}_{L^2_3}$$

thus $Z(\kappa_A)$ is diffeomorphic to a neighbourhood of $[A]$. □

For $A \in \mathcal{M}_P$, not necessarily irreducible,

then $\mathbb{H}^1_A(*)$, $\mathbb{H}^2_A(*)$ are Γ_A-invariant,

and κ : Γ_A-equivariant,

and $\mathcal{M}_P$ is locally modelled around $[A]$ on

$$\kappa_A^{-1}(0) / \Gamma_A .$$

<u>Rmk</u> <u>$\mathbb{H}^0_A(*) = 0$</u>,

then $\dim \mathbb{H}^1_A(*) - \dim \mathbb{H}^2_A(*) = \text{const}$.

but $\dim \mathbb{H}^2_A(*)$ may vary.

$\Longrightarrow$ derived manifold.

<u>$\mathbb{H}^0_A(*) \neq 0$</u>

$\exists$ quotient singularity.

$\Longrightarrow$ derived stacks

§5. Freed-Uhlenbeck generic smoothness

Let $\mathcal{C}$: the space of conformal structures on X

(i.e. $\{\text{metrics}\}/\text{conf. transf}$)

complete this by e.g. C^r $(r>3)$ norm.

to get a Banach manifold.

Thm (Freed-Uhlenbeck)

$G = SU(2)$, $SO(3)$, $\pi_1(X) = \{1\}$.

$\exists$ open dense subset $\mathcal{D} \subset \mathcal{C}$ such that

if $[g] \in \mathcal{D}$, then for any $[A] \in \mathcal{M}_P^*$,

$\mathbb{H}^2_A (*) = 0$

Sketch of proof

Consider $\mathcal{E} := \mathcal{A}_P^* \times_{\mathcal{G}_P / C(P)} \Omega^+(\mathrm{ad}(P)) \to \mathcal{B}_P^*$, $\quad s = F_A^+$

(fibre $= \Omega^2(\mathrm{ad}(P))$)

$$s^{-1}(0) = \mathcal{M}_P^*$$

We'd like to perturb s to get its derivative being surjective on the zero set.

In order to do this, consider

$$\pi^*(\mathcal{E}) \to \mathcal{B}_P^* \times \mathcal{C}, \quad \hat{s}$$

where $\pi : \mathcal{B}_P^* \times \mathcal{C} \to \mathcal{C}$, projection

$\hat{s}$: extension of s

(it's Fredholm in the direction of $\mathcal{B}_P^*$)

<u>Prop</u>. $G = SU(2), SO(3)$, $\pi_1(X) = \{1\}$.

$\Rightarrow$ the derivative of $\hat{s}$

$$D\hat{s} : T\mathcal{B}_P^* \times \mathcal{C} \to \Omega_X^+(\mathrm{ad}(P))$$

is surjective on the zero set of S.

Call this property <u>regular</u>.

<u>Fact</u> (a version of Sard-Smale)

$\mathcal{V} \to \mathcal{P}$, a bundle of Banach space over a Banach manifold.

Φ : Fredholm section

$\pi^*(\mathcal{V}) \to \mathcal{P} \times \mathcal{S}$, $\mathcal{S}$: Banach manifold.

$\hat{\Phi}$: extension of Φ, Fredholm in the $\mathcal{P}$-direction

If $\mathcal{Z} = \hat{\Phi}^{-1}(0)$ is regular.

$\Rightarrow \exists$ a dense (second category) set of parameters $s \in \mathcal{S}$

for which $\hat{\Phi}_s^{-1}(0)$ is regular. $\square$